

2.7c Liapunov exponents

Monday, February 8, 2021 11:48 AM

Def. 2.7 Let $f: I \rightarrow I$. Then the map f has **sensitive dependence** on the initial condition x_0 if $\exists \delta > 0$ s.t. $\forall \epsilon > 0$, $\exists y_0 \in I$ and $k \in \mathbb{N}$ s.t.

$$|x_0 - y_0| < \epsilon \quad \text{and} \quad |f^k(x_0) - f^k(y_0)| > \delta.$$

For a given initial condition x_0 ,

$$\text{let } \Delta(\epsilon, t) = |f^t(x_0 + \epsilon) - f^t(x_0)|$$

Suppose $x_{t+1} = 2x_t$ as a simple example, with $x_0 = 0$.

$$\text{Then } \Delta(\epsilon, t) = |2^t(x_0 + \epsilon) - 2^t x_0| = |2^t \epsilon| = |\epsilon e^{t \ln 2}|$$

- $\Delta(\epsilon, t)$ should have linear dependence on ϵ
- $\Delta(\epsilon, t)$ should have exponential dependence on t .

Thus, we can consider defining the **Liapunov exponent** $\lambda(x_0)$ by

$$\epsilon e^{t \lambda(x_0)} \approx |f^t(x_0 + \epsilon) - f^t(x_0)|, \text{ where } \epsilon \text{ is small and } t \text{ is large}$$

$$\text{Let } \epsilon \rightarrow 0 \quad e^{t \lambda(x_0)} \approx \lim_{\epsilon \rightarrow 0} \left| \frac{f^t(x_0 + \epsilon) - f^t(x_0)}{\epsilon} \right|$$
$$= \left| \frac{d}{dx} f^t(x_0) \right|$$

$$\Rightarrow \lambda(x_0) = \frac{1}{t} \ln \left| \frac{d}{dx} f^t(x_0) \right|$$
$$= \frac{1}{t} \ln |f'(x_0) f'(x_1) \cdots f'(x_{t-1})|$$
$$= \frac{1}{t} \sum_{k=0}^{t-1} \ln |f'(x_k)|.$$

But still have t -dependence, so let $t \rightarrow \infty$

$$\lambda(x_0) = \lim_{t \rightarrow \infty} \frac{1}{t} \sum_{k=0}^{t-1} \ln |f'(x_k)|.$$

Def 2.8 The Liapunov exponent (Lyapunov exponent) at x_0 of $x_{t+1} = f(x_t)$ is

$$\lambda(x_0) = \lim_{t \rightarrow \infty} \frac{1}{t} \sum_{k=0}^{t-1} \ln |f'(x_k)|$$

If $\lambda(x_0)$ is independent of x_0 , then $\lambda(x_0)$ is denoted by λ and referred to as the Lyapunov exp of f . There exist sensitive dependence on initial conditions if $\lambda(x_0) > 0$ for all initial conditions in the domain.

Aside: Can verify all stable equilibria have negative Lyapunov exponents
unstable equilibria have positive Lyapunov exponents.

In practice: The Lyapunov exponents for nonequilibria are harder to compute, so we often approximate them. Note, for fixed j , $\lambda(x_0) = \lambda(x_j)$,

$$\lambda(x_0) \approx \frac{1}{N} \sum_{k=j}^{k=j+N-1} \ln |f'(x_k)|$$

Aside: We can generalize to system of first-order equations

$$X(t+1) = F(X(t))$$

by using the Jacobian matrix $J(X(t))$, and looking at the spectral radius ρ .

Lyapunov exp $\lambda(X(0)) = \lim_{t \rightarrow \infty} \ln \rho \left(\prod_{k=0}^{t-1} J(X(k)) \right)$.

If $\lambda(X(0)) > 0$, then the solution to the system exhibits sensitive dependence on the initial condition $X(0)$.

If the solution is not asymptotically periodic, and no

If the solution is not asymptotically periodic, and no Lyapunov exp is exactly 0, then the solution is chaotic.